Nonembedness of the Klein Bottle in $\mathbb{RP}^3$
and the Lawson’s conjecture

ABSTRACT: In 1985 Montiel & Ros showed that the only minimal torus in $S^3$, which first eigenvalue of the Laplacian is 2, is the Clifford torus. Here, we will show first the no existence of an embedded Klein bottle in $\mathbb{RP}^3$, indeed we will prove that the only non orientable compact surfaces that can be embedded in $\mathbb{RP}^3$ are those with odd Euler characteristic. Later on, we will show another proof of Montiel & Ros’ result but assuming that the minimal torus has $\{x, -x\}$ symmetry. We will also point out that our proof of the no existence of embedded closed non orientable surfaces in $\mathbb{RP}^3$ with even Euler characteristic, still holds true when we replace $\mathbb{RP}^3$ for a 3 dimensional manifold $K$ constructed in the following way: Let $N$ be any compact, simply connected 3 dimensional manifold. Let $f : S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3\} \to N$ be an embedding. Let $U$ and $V$ be the two connected components of $N \setminus f(S^2)$. $K$ is the manifold obtained by taking $U$, and identifying the points in $\partial U$ so that $f(x) = f(-x)$.

§1. Introduction: It is well known that it is impossible to embed a nonorientable compact surface in $\mathbb{R}^3$, [S], however it is possible to embed a projective plane, $\mathbb{RP}^2$, in $\mathbb{RP}^3$. Notice that $\mathbb{RP}^2$ is a non orientable surface while $\mathbb{RP}^3$ is orientable. In the first part of this paper we prove, in a constructive and simple way, that we can not embed neither the Klein bottle nor a Klein bottle with a finite number of handles attached in $\mathbb{RP}^3$.

Minimal hypersurfaces of spheres have been a subject of great importance. They represent critical points of a variational problem and also the study of these hypersurfaces is related with the regularity of the Plateau problem. A first step in this study was to consider surfaces in $S^3$. The easiest examples of minimal surfaces are the equators, which are surfaces isometric to the set

$$\{(x, y, z, w) \in \mathbb{R}^4 : w = 0 \quad \text{and} \quad x^2 + y^2 + z^2 = 1\}$$

and the Clifford torus, which are surfaces isometric to the set

$$\{(x, y, z, w) \in \mathbb{R}^4 : z^2 + w^2 = \frac{1}{2} \quad \text{and} \quad x^2 + y^2 = \frac{1}{2}\}$$

In 1966, Almgren showed, [A], that the only immersed minimal spheres in $S^3$ are the equators. Even though there are infinitely many ways to minimally immerse a torus in $S^3$, the only known example that is embedded is the Clifford torus. The conjecture that asserts that the Clifford torus is the only embedded minimal torus in $S^3$ is known as the Lawson’s conjecture. It is not difficult to prove that for every immersed compact minimal surface in $S^3$, 2 is an eigenvalue of the laplacian operator. One of the well known conjectures in the study of minimal hypersurfaces of spheres is Yau’s conjecture, this conjecture, in the case of surfaces, states that if a compact surface in $S^3$ is embedded and minimal, then 2 is the first eigenvalue of the Laplacian. Montiel & Ros, [M-R], showed that for minimal torus in $S^3$, Yau’s conjecture implies Lawson’s conjecture. In the second part of this paper, we will use the main theorem of the first part to prove Montiel & Ros’ result in a shorter way under the additional hypothesis that the minimal torus has antipodal symmetry.
§2. Preliminaries: In this section we will establish some results that we will use to prove our main theorems. Let us start with transversality theory. From linear algebra is well known that, in general, the intersection of two 2 dimensional subspaces in \( \mathbb{R}^3 \) is a 1 dimensional space. When we have two surfaces, \( M_1 \) and \( M_2 \), in a 3 dimensional manifold \( N \), we have, by the implicit function theorem, that if these surfaces satisfy that

\[
T_m M_1 \cap T_m M_2 \subset T_m N \quad \text{is 1 dimensional for every } m \in M_1 \cap M_2 \tag{1}
\]

then the set \( M_1 \cap M_2 \) is either empty or it is a 1 dimensional submanifold of \( N \).

When the condition (1) holds true, we say that \( M_1 \) intersects transversally to \( M_2 \).

A theorem in Tranversality theory gives us,

**Theorem 2.1:** Given two smooth surfaces \( S_1 \) and \( M_2 \) in a 3 dimensional manifold \( N \), it is possible to find a smooth surface \( M_1 \subset N \) such that \( S_1 \) is homeomorphic to \( M_1 \) and \( M_1 \cap M_2 \) is either empty or a 1 dimensional manifold in \( N \).

Let us state the following theorem on quotient manifold,

**Theorem 2.2:** Let \( G \times M \rightarrow M \) be a properly discontinuous action of a group \( G \) on a differentiable manifold \( M \). The manifold \( M/G \) is orientable if and only if there is an orientation of \( M \) that is preserved by all the diffeomorphisms of \( G \).

As a consequence of this theorem we have the following examples: Let us denote by \( S^n = \{ x \in \mathbb{R}^{n+1} : |x| = 1 \} \)

**Example 2.1:** Let \( G = \{-1,1\} \) act on \( S^n \) by \((-1,x) \rightarrow -x \) and \((1,x) \rightarrow x \). Clearly this action is properly discontinuous. The diffeomorphism \(-1 \) sends the bases \( \{v_1,\ldots,v_n\} \) of \( T_m S^n \) to the bases \( \{-v_1,\ldots,-v_n\} \) of \( T_{-m} S^n \). Let us assume that we are taking the orientation on \( S^n \) given by the normal unit vector field \( \nu(m) = m \). Using this orientation, we have that if \( \{v_1,\ldots,v_n\} \) is an oriented bases of \( T_m S^n \), then the same bases does not provide an oriented bases of \( T_{-m} M \). Therefore, the diffeomorphism \(-1 \), which sends the orientation given by a bases \( \{v_1,\ldots,v_n\} \) of \( T_m S^n \) to the orientation given by the basis \( \{-v_1,\ldots,-v_n\} \) of \( T_{-m} M \), reverses the orientation on \( S^n \) if and only if \( n \) is even. Hence \( \mathbb{RP}^n = S^n/\{-1,1\} \) is orientable if and only if \( n \) is odd.

**Example 2.2:** Let \( M \) be an embedded torus in \( S^3 \) such that if \( m \in M \) then \( -m \in M \). Let \( \nu : M \rightarrow S^3 \) be a normal vector field of \( M \) as a submanifold of \( S^3 \), i.e. \( \nu(m) \) is perpendicular to \( T_m M \) and \( \nu(m) \) is a vector in \( T_m S^3 \). Since \( M \) has antipodal symmetry, then \( T_m M = T_{-m} M \) for every \( m \in M \). Therefore we have that either \( \nu(m) = \nu(-m) \) for all \( m \in M \) or \( \nu(-m) = -\nu(m) \) for all \( m \in M \). As we pointed out before, the vector spaces \( T_m S^3 \) and \( T_{-m} S^3 \) are the same but they have different orientation; this implies that if \( \nu(m) = \nu(-m) \) then the orientation induced by \( M \) on \( T_m M \) and \( T_{-m} M \) is also different. Since the basis \( \{v_1,v_2\} \) and \( \{-v_1,-v_2\} \) induces the same orientation on the vector space \( \{rv_1 + sv_2 : r,s \in \mathbb{R}\} \), we have that if \( \nu(m) = \nu(-m) \), then the manifold \( S = M/\{-1,1\} \) is not orientable. Since the Euler characteristic of \( M \), \( \chi(M) \), is two times the Euler characteristic of \( S = M/\{-1,1\} \) and \( \chi(M) \) is zero, then \( S = M/\{-1,1\} \) is a
Klein bottle. The same argument shows that if $\nu(-m) = -\nu(m)$ then $S = M/\{-1,1\}$ is again a torus.

§3. Nonembedness of the Klein bottle in $\mathbb{RP}^3$

In this section, we will prove that it is impossible to embed a Klein bottle or a Klein bottle with a finite number of handles attached in the 3-dimensional projective space $\mathbb{RP}^3$. We will achieve this by using some basic criteria to decide when a surface is orientable and by making some constructions in order to estimate de Euler characteristic of any embedded surface in $\mathbb{RP}^3$.

Let us identify $\mathbb{RP}^3$ with the set $N$ of points in $\mathbb{R}^3$ with norm less than or equal to 1 where every point in the boundary is identified with its opposite, i.e. if

\[
B = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 \leq 1\}
\]

\[
\partial B = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}
\]

\[
\tau : \partial B \to \partial B \quad \tau(m) = -m
\]

then

\[
N = B/\{id, \tau\}
\]

We may think that we are identifying $N$ with $\mathbb{RP}^3$ using the map

\[
\phi : N \to \mathbb{RP}^3 \quad \text{given by} \quad \phi(x_1, x_2, x_3) = [(x_1, x_2, x_3, \sqrt{1-x_1^2-x_2^2-x_3^2})]
\]

Clearly $\phi$ is bijective because the antipodal points at the boundary of $B$ are identified.

Notice that under this identification, $\mathbb{RP}^2$ is identified with $\partial B/\{id, \tau\} \subset N$

Let us denote by $\pi : B \to N$ the natural projection, i.e. $\pi(x) = x$ if $|x| < 1$ and $\pi(x) = [x] = \{x, -x\}$ if $|x| = 1$.

**Lemma 3.1:** If $M \subset N$ is an embedded surface that intersects transversally $\mathbb{RP}^2$, then the set $C_1 = \pi^{-1}(M \cap \mathbb{RP}^2) \subset S^2 = \partial B$ has one of the following forms

\[
C_1 = \{\alpha_0, \alpha_1, \ldots, \alpha_k, \overline{\alpha}_k\}
\]

\[
C_1 = \{\alpha_1, \overline{\alpha}_1, \ldots, \alpha_k, \overline{\alpha}_k\}
\]

where the $\alpha_i$ and $\overline{\alpha}_i$ is a family of disjoint closed curves with $\tau(\alpha_i) = \overline{\alpha}_i$
Proof: Let $M_1 = \pi^{-1}(M) \subset B$. Notice that $C_1$ is a collection of smooth disjoint closed embedded curves in $\partial B$ because $(M \cap \mathbb{RP}^2)$ is a collection of smooth closed disjoint curves and the map $\pi_{|S^2} : S^2 = \partial B \to \mathbb{RP}^2$ is a covering map. Notice also that $M_1$ is an embedded surface with boundary in $\mathbb{R}^3$, moreover $\partial M_1 = C_1$. By the identification made on $N$ we have that if $x \in C_1$ then $-x \in C_1$. Let us prove by contradiction that there is at most one closed curve contained in $C_1$ that have antipodal symmetry. Let $\alpha_0$ and $\alpha'_0$ be two disjoint closed curves contained in $C_1$, since $\alpha_0$ is an embedded curve in $S^2$, it divides $S^2$ in two simply connected parts $U$ and $V$; since $\alpha_0$ has antipodal symmetry, then $\tau(U) = V$, therefore the area of $U$ is the same as the area of $V$ and both area are equal to $2\pi$ because the area of the $S^2$ is $4\pi$; now, since $\alpha_0$ and $\alpha'_0$ are disjoint, then one of the connected components of $S^2 - \alpha'_0$, let us call it $W$, is contain in either $U$ or $V$, this is a contradiction because the area of $W$ is $2\pi$. Since there can only be one closed curve with antipodal symmetry in $C_1$ we have that there are two possibilities for the set $C_1$

Case 1: If $C_1$ contains a circle $\alpha_0$ which is invariant under $\tau$, then

$$C_1 = \{\alpha_0, \alpha_1, \bar{\alpha}_1, \ldots, \alpha_k, \bar{\alpha}_k\}$$

where the $\alpha_i$'s are closed curves, $\tau(\alpha_0) = \alpha_0$ and $\tau(\alpha_i) = \bar{\alpha}_k$ for $i = 1, 2, \ldots, k$.

Case 2: If $C_1$ does not contain a circle which is invariant under $\tau$, then

$$C_1 = \{\alpha_1, \bar{\alpha}_1, \ldots, \alpha_k, \bar{\alpha}_k\}$$

where the $\alpha_i$'s are closed curves, and $\tau(\alpha_i) = \bar{\alpha}_k$ for $i = 1, 2, \ldots, k$.

Theorem 3.1 If $M$ is a compact surface in $\mathbb{RP}^3$ that intersects transversally $\mathbb{RP}^2$ and $C_1 = \pi^{-1}(M \cap \mathbb{RP}^2) \subset S^2 \subset \partial B$ contains a closed curve which is invariant under the antipodal map, then the Euler characteristic of $M$ is odd.

Proof: In the same way we did before, let us identify $\mathbb{RP}^3$ with $N = B/\{id, \tau\}$ and $\mathbb{RP}^2$ with $\partial B/\{id, \tau\}$. Since there is a closed curve in $C_1$ invariant under the antipodal map, we have, by Lemma 3.1, that

$$C_1 = \{\alpha_0, \alpha_1, \bar{\alpha}_1, \ldots, \alpha_k, \bar{\alpha}_k\} = \{\beta_0, \beta_1, \ldots, \beta_{2k}\}$$

with $\tau(\alpha_i) = \alpha_i$.

For $i = 1, \ldots, 2k$, let $B_i$ be connected component with smaller area of $S^2 \setminus \beta_i$. Let $M_2$ be the manifold that is obtained by gluing to $M$ a $2k + 1$ disks, one along each $\beta_i$. It is not difficult to see that we can embed $M_2$ in $\mathbb{R}^3$, e.g. we can make this gluing in $\mathbb{R}^3$ by choosing disks of the form

$$\{rx : 1 \leq r \leq r_i \text{ and } x \in \beta_i\} \cup \{r_i x : x \in B_i\}$$

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for $i = 1, \ldots, 2k$, and the last disk of the form

$$\{rx : 1 \leq r \leq r_0 \text{ and } x \in \beta_0\} \cup \{r_0 x : x \in V\}$$

where $V$ is one of the connected components of $S^2 \setminus \alpha_0$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Figure 1}
\end{figure}

In this figure $M \subset \mathbb{RP}^3$ is homeomorphic to $\mathbb{RP}^2$ and it intersects $\partial B/\{-1, 1\}$ in two circles. The inverse image of these circles under $\pi : B \to \mathbb{RP}^3$ is the union of three circles, one of them, the equator, is invariant under the antipodal map. $M_1 = \pi^{-1}(M)$ is the union of a cylinder and a disk, $M_2$ is the union of two closed spheres. In this case $r_1 = r_2 = 1$, $r_0 > 1$ and $V$ is the souther hemisphere.

Let us denote by $\chi(S)$ the Euler characteristic of a surface $S$. Let us take a triangulation of $M$ such that each circle $\pi(\alpha_i) \subset M$ contains exactly 3 edges of the triangulation. Let $F$ be the number of faces, $E$ the number of edges and $V$ the number of vertices of the triangulation. Clearly this triangulation induces a triangulation on $M_1 = \pi^{-1}(M)$, the number of faces, edges and vertices for this triangulation on $M_1$ is $F$, $E + 3k + 3$ and $V + 3k + 3$, respectively, this happens because the circle $\alpha_0$ contains now 6 edges and 6 vertices instead of the 3 edges and 3 vertices of $\pi(\alpha_0)$ and, for $i = 1, \ldots, k$, the 3 edges and 3 vertices on $\pi(\alpha_i)$ give us 3 edges and 3 vertices in $\alpha_i$ and $\overline{\alpha}_i$. Therefore, $\chi(M) = \chi(M_1)$.

Now taking this triangulation on $M_1$, we define a new triangulation on $M_2$ by adding $6 + 2k$ new triangles to the triangulation defined on $M_1$ in this way:

(i) The disk attached to the circle $\alpha_0$ is thought as 6 triangles with 6 vertices in the boundary and one vertex in the interior of the disk, the gluing is taken so that vertices on the boundary of the glued disk are identified with the 6 vertices of $\alpha_0$. 


(ii) Each disk attached to an $\alpha_i$ (or $\bar{\alpha}_i$) is thought just as a single triangle and the gluing is taken so that the 3 vertices of the boundary of the glued disk are identified with the 3 vertices of $\alpha_i$ (or $\bar{\alpha}_i$).

Having made these considerations, it is not difficult to set that the new triangulation is going to have $6 + 2k$ more faces, 6 more edges and 1 more vertex than the triangulation on $M_1$, therefore,

$$\chi(M_2) = \chi(M_1) + 2k + 1 = \chi(M) + 2k + 1$$

Since $M_2$ can be embedded in $\mathbb{R}^3$ we have that $M_2$ is orientable, henceforth, its Euler characteristic is even, this observation, together with the equation above implies that the Euler characteristic of $M$ must be odd.

**Theorem 3.2** If $M$ is a compact surface in $\mathbb{RP}^3$ that intersects transversally $\mathbb{RP}^2$ and $C_1 = \pi^{-1}(M \cap \mathbb{RP}^2)$ does not contain a circle which is invariant under the antipodal map, then $M$ is orientable.

**Proof:** Since none of the closed curves in $C_1$ is invariant under $\tau$, then

$$C_1 = \{\alpha_1, \bar{\alpha}_1, \ldots, \alpha_k, \bar{\alpha}_k\}$$

with $\tau(\alpha_i) = \bar{\alpha}_i$. Let us assume that $k = 1$. Let $K_1^\epsilon$ be the set of points in $M_1 = \pi^{-1}(M)$ that are within a distance $\epsilon$ of $\alpha_1$ and let $K_2^\epsilon$ be the set of points in $M_1$ that are within a distance $\epsilon$ of $\bar{\alpha}_1$, we will assume that $\epsilon$ has been chosen so that $K_i^\epsilon$ are smooth surfaces homeomorphic to cylinders. Let $M_2$ be the compact surface obtained by gluing to $M_1$ a cylinder $\Sigma$ embedded in $\mathbb{R}^3 \setminus B$, the boundary of this glued cylinder is $\alpha_1 \cup \bar{\alpha}_1$. Let $\phi : [0, 1] \to \alpha_1$ be a regular parametrization of $\alpha_1$. Notice that the map $\phi$ is homotopic to the map $\tau \circ \phi$ in $\Sigma$ (this is the key observation in this proof). Therefore we can define a parametrization

$$\psi : [0, 1] \times [0, 1] \to \Sigma \text{ such that }$$

$$\psi(0, v) = \psi(1, v)$$

$$\psi(u, 0) = \phi(u) \text{ and }$$

$$\psi(u, 1) = \tau \circ \phi(u)$$

We will prove that $M$ is homeomorphic to $M_2$. This would imply that $M$ is orientable because $M_2$ is orientable, recall that a compact surfaces can be embedded in $\mathbb{R}^3$ only if it is orientable. Let $\Sigma_1 = \phi([0, 1] \times [0, 1/2])$ and $\Sigma_2 = \phi([0, 1] \times [1/2, 1])$. Let $\gamma_i$, $i = 1, 2$ be two closed curves in $M_1$ such that $\partial K_1^\epsilon = \alpha_1 \cup \gamma_1$ and $\partial K_2^\epsilon = \bar{\alpha}_1 \cup \gamma_2$. Since $\Sigma_i$ is homeomorphic to $K_i$, then, we can define two homeomorphisms $\varphi_1$ and $\varphi_2$ such that
\[ \varphi_1 : \Sigma_1 \rightarrow K_1^\epsilon \quad \varphi_2 : \Sigma_2 \rightarrow K_2^\epsilon \]

\[ \varphi_1(\psi(u, 1/2)) = \psi(u, 0) = \phi(u) \quad \varphi_2(\psi(u, 1/2)) = \psi(u, 1) = \tau \circ \phi(u) \]

\[ \varphi_1(\psi(u, 0)) \in \gamma_1 \quad \varphi_2(\psi(u, 1)) \in \gamma_2 \]

**Figure 2**

In this figure \( M \subset \mathbb{R}P^3 \) is homeomorphic to a double torus and it intersects \( \partial B / \{-1, 1\} \) in one circle. The inverse image of these circles under \( \pi : B \rightarrow \mathbb{R}P^3 \) is the union of two circles. \( M_1 = \pi^{-1}(M) \) is a cylinder with a handle attached. \( M_2 \) is again homeomorphic to a double torus.

Since the manifold \( M_1 \) and \( M_1 \setminus K_1^\epsilon \cup K_2^\epsilon \) are homeomorphic we can define a homeomorphism \( \varphi_3 \) such that

\[ \varphi_3 : M_1 \rightarrow M_1 \setminus K_1^\epsilon \cup K_2^\epsilon \quad \text{such that} \]

\[ \varphi_3(\psi(u, 0)) = \varphi_1(\psi(u, 0)) \quad \text{and} \]

\[ \varphi_3(\psi(u, 1)) = \varphi_2(\psi(u, 1)) \]

Using the maps \( \varphi_1 \), \( \varphi_2 \) and \( \varphi_3 \), we can define our homeomorphism \( \xi \) from \( M_2 \) to \( M \) in the following way:

\[ \xi(m) = [\varphi_1(m)] \quad \text{if} \ m \in \Sigma_1 \]

\[ \xi(m) = [\varphi_2(m)] \quad \text{if} \ m \in \Sigma_2 \]

\[ \xi(m) = [\varphi_3(m)] \quad \text{if} \ m \in M_1 \]
The map $\xi$ is a continuous well defined map because of the the conditions imposed on the maps $\varphi_i$, $i = 1, 2, 3$ on the boundary. Therefore $M$ is homeomorphic to $M_2$ which is orientable. When $k > 1$, let $B_i$ be the connected component with smaller area of $S^2 \setminus \alpha_i$ for $i = 1, \ldots, k$. In the case the sets $B_1, \ldots, B_k$ are disjoint, let us take $k$ disjoint curves, $\omega_1, \ldots, \omega_k$, contained in the closure of the set $\mathbb{R}^3 \setminus B$ such that each $\omega_i$ connects a point $p_i \in B_i$ with the point $-p_i$. Let $M_2$ be the surface obtained by gluing to $M_1 = \pi^{-1}(M)$ $k$ cylinders $\Sigma_1, \ldots, \Sigma_k$. This cylinder are chosen so that the boundary of each $\Sigma_i$ is the union of $\alpha_i$ with $\overline{\pi}_i$ and the bounded component of the compact surface $\Sigma_i \cup B_i \cup \tau(B_i)$ contains the curve $\omega_i$, in other words, $\Sigma_i$ is the cylindrical part of the boundary of the solid obtained after thickening the curve $\omega_i$. The same procedure that we did in the case $k = 1$ shows that the surface $M$ is homeomorphic to the surface $M_2$, since $M_2$ is embedded in $\mathbb{R}^3$, then $M_2$ is orientable, hence $M$ is also orientable too. The proof in the general case is essentially the same, we glue $\Sigma_1, \ldots, \Sigma_k$ cylinders to the surface $M_1$ to obtain an orientable surface that is homeomorphic to $M$. In this general case, the cylinders can be chosen so that every time $B_i \subset B_j$ then, $\Sigma_i$ is contained in the bounded component of the compact surface $\Sigma_j \cup B_j \cup \tau(B_i)$.

**Theorem 3.3:** A compact non oriented surface $M$ can be embedded in $\mathbb{RP}^3$ if and only if the Euler characteristic of $M$ is odd.

**Proof:** The set $M = \{([x, y, 0]) : x^2 + y^2 \leq 1\} \subset N = B/\{id, \tau\} = \mathbb{RP}^3$ shows an embedding of $\mathbb{RP}^2$ in $\mathbb{RP}^3$, clearly we can attach as many handles as we want in a embedded fashion to $M$ in a neighborhood of the point $(0,0,0)$. This shows that every non oriented closed surface with odd Euler characteristic can be embedded in $\mathbb{RP}^3$. Now, Let $S$ be a non orientable surface embedded in $\mathbb{RP}^3$. By Theorem 2.1, there exists an embedded surface $M \subset \mathbb{RP}^3$ homeomorphic to $S$, that intercepts transversally $\mathbb{RP}^2 = \partial B/\{-1, 1\}$, by Lemma 3.1, Theorem 3.1 and Theorem 3.2, we have that the Euler characteristic of $M$ must be odd, therefore the Euler characteristic of $S$ must be also odd.

**Corollary 3.1:** It is impossible to embed a Klein bottle, or any compact non oriented compact surface with even Euler characteristic, in $\mathbb{RP}^3$.

Using the same method that we used to prove Theorem 3.3, we can prove the slightly more general result:

**Theorem 3.4** Let $N$ be any compact, simply connected 3 dimensional manifold. Let $f : S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3\} \to N$ be an embedding. Let $U$ and $V$ be the two connect components of $N \setminus f(S^2)$. If $K$ is the manifold obtained by taking $U$, and identifying the points in $\partial U$ so that $f(x) = f(-x)$, then a compact non oriented surface $M$ can be embedded in $K$ if and only if the Euler characteristic of $M$ is odd.

**Proof:** Since the intersection of any closed curve in $N$ that intercepts transversely $f(S^2)$ must be even, $[S]$, we have that, as the statement of the theorem suggests, $N \setminus f(S^2)$ has exactly two connected components. Let $K_1$ be the manifold obtained by gluing the unit ball $B$ in $\mathbb{R}^3$ to $U$ using the map $f$. All the arguments made to prove Theorem 3.1 to Theorem 3.3 work by replacing $\mathbb{R}^3$ by the simply connected manifold $K_1$, and $\mathbb{R}^3 \setminus B$ by the set $B$ view as a subset of $K_1$. Notice also that every non orientable surface embedded in $K$ must
intersect the surface $\mathbb{RP}^2 = f(S^2)/\{1, \tau\}$ where $\tau$, in this case, is the map from $f(S^2)$ to $f(S^2)$ defined by $\tau(p) = f(-f^{-1}(p))$. \[\square\]

§4. A partial result on the Lawson’s conjecture

Let us start this section with some well known facts on minimal surfaces on $S^3$. If $\psi : M \to S^3$ is an immerse minimal surface in $S^3$ and $\psi(m) = (x_1(m), x_2(m), x_3(m), x_4(m))$ then the minimallity of $M$ is equivalent to the condition $\Delta x_i = -2x_i$ on the four functions $x_i : M \to S^3, i = 1, \ldots, 4$. Let us denote by $\nu : M \to S^3$, $\nu(m) = (\nu_1(m), \nu_2(m), \nu_3(m), \nu_4(m))$, be the unit normal vector field on $M$ as a submanifold of $S^3$. The shape operator at $m \in M$ is the symmetric linear operator $A_m : T_mM \to T_mM$ defined by $A_m(v) = -d\nu_m(v)$. By the Codazzi equations and the minimallity of $M$, we have that the functions $\nu_i : M \to S^3$ satisfy the equation $\Delta \nu_i = -|A|^2\nu_i$ for $i = 1, \ldots, 4$. Here $|A|^2(m) = |A_m(e_1)|^2 + |A_m(e_2)|^2$, where $\{e_1, e_2\}$ is any orthonormal basis of $T_mM$. Notice that,

$$|A|^2(m) = |A_m(e_1)|^2 + |A_m(e_2)|^2 = |d\nu_m(e_1)|^2 + |d\nu_m(e_2)|^2 = |(d(\nu_1)_m(e_1), d(\nu_2)_m(e_1), d(\nu_3)_m(e_1), d(\nu_4)_m(e_1))|^2 + |(d(\nu_1)_m(e_2), d(\nu_2)_m(e_2), d(\nu_3)_m(e_2), d(\nu_4)_m(e_2))|^2 = \sum_{i=1}^4 (d(\nu_i)_m(e_1)^2 + d(\nu_i)_m(e_2)^2) = \sum_{i=1}^4 \|\nabla \nu_i\|^2(m)$$

The eigenvalues $\kappa_1(m), \kappa_2(m)$ of $A_m$ are known as the principal curvatures of $M$ at $m \in M$. The Gauss equation applied to surfaces in $\mathbb{R}^3$ gives us that the Gauss curvature is the product of the principal curvatures. For surfaces in $S^3$ the Gauss equation gives us that the Gauss curvature is the product of the principal curvatures plus 1, i.e. if $K(m)$ is the Gauss curvature of $M \subset S^3$ at $m \in M$, we have

$$K(m) = 1 + \kappa_1(m)\kappa_2(m) \tag{2}$$

Since $M$ is minimal, then $\kappa_1(m) + \kappa_2(m) = 0$, then $|A|^2(m) = \kappa_1^2(m) + \kappa_2^2(m) = 2\kappa_1^2(m)$, therefore the equation (2) can be written as

$$K(m) = 1 - \frac{|A|^2}{2}$$

In the case that $M$ is topologically a torus, the Gauss Bonnet formula gives us that $\int_M K = 0$ or equivalently

$$9$$
\[ \int_M |A|^2 = \int_M 2 = 2 \text{ times the area of } M \]  

(3)

Now we are ready to prove the main theorem in this section.

**Theorem 4.1:** If \( M \) is an embedded minimal surface in \( S^3 \) which is invariant under the antipodal map and such that the first eigenvalue of the Laplacian is 2, then \( M \) is the Clifford torus.

**Proof:** Let \( \nu = (\nu_1, \nu_2, \nu_3, \nu_4) \) be the unit normal vector. By Corollary 3.1 and Example 2.2 we have that \( \nu(-m) = -\nu(m) \) for all \( m \in M \) otherwise \( M/\{-1,1\} \) will define an embedded Klein bottle in \( \mathbb{RP}^3 \). Therefore the function \( \nu_i : M \to \mathbb{R}, \ i = 1, \ldots, 4 \) are odd functions and \( \int_M \nu_i = 0 \) i.e. they are functions perpendicular to the constant function \( f \equiv 1 \) viewed as elements of the Hilbert space \( L^2(M) \). Now, since we are assuming that the first eigenvalue of the Laplacian is 2, we have

\[ \int |\nabla \nu_i|^2 \geq 2 \int_M \nu_i \quad i = 1, 2, 3, 4 \]  

(4)

with equality if and only if

\[ \Delta \nu_i = -|A|^2 \nu_i = -2\nu_i \]  

(5)

Notice that if the equation (5) holds true for \( i = 1, \ldots, 4 \), then \( |A|^2(m) = 2 \) for all \( m \in M \). Summing up the inequalities in (4) above from \( i = 1 \) to \( i = 4 \) we get

\[ \int |A|^2 \geq \int_M 2 \]  

(6)

but the equation (3) gives us that we have an equality in the equation (6) instead of an inequality, therefore, we have an equality in each of the equations in (4), hence, \( |A|^2(m) = 2 \) for all \( m \in M \). This last equation implies that \( M \) must be a Clifford torus by the main result in [C-D-K].

**REFERENCES**


