Minimal hypersurfaces in $\mathbb{R}^n$ as regular values of a function

Oscar Perdomo
Universidad del Valle, Cali - Colombia

ABSTRACT: In this paper we prove that if $M = f^{-1}(0)$ is a minimal hypersurface of $\mathbb{R}^n$, where $f : V \subset \mathbb{R}^n \to \mathbb{R}$ is a smooth function defined on an open set $V$, then $f$ must satisfy the equation $|\nabla f|^2 \Delta f = \frac{1}{2} \langle |\nabla f|^2, \nabla f \rangle$ for every $x \in M$. We will also prove that if $M$ is the zero level set of a homogeneous 2 polynomial, then $M$ must be a Clifford minimal hypersurface.

§1 Introduction and preliminaries:
In this paper we will consider hypersurfaces $M \subset \mathbb{R}^n$ that are level sets of functions, i.e. we will assume that $M = \{x \in V : f(x) = 0\}$ where $f : V \to \mathbb{R}$ is a smooth function defined in an open set of $\mathbb{R}^n$ and $|\nabla f(x)| \neq 0$ for all $x \in M$. For these hypersurfaces, we have that the Gauss map can be written as $\nu(x) = \frac{1}{|\nabla f(x)|} \nabla f(x)$ for all $x \in M$. Clearly, the tangent space of $M$ at a point $x$ is the space of vectors $v \in \mathbb{R}^n$ such that $\langle v, \nabla f(x) \rangle = 0$. Notice that the mean curvature of $M$ at $x$ is given by

$$-\sum_{i=1}^{n-1} \langle d\nu_x(v_i), v_i \rangle$$

where $\{v_1, \ldots, v_{n-1}\}$ is an orthonormal bases of the vector space $T_xM$.

I would like to mention some elementary facts about real value functions on $\mathbb{R}^n$ that will be used later on.

Lemma 1.1: If $f : V \to \mathbb{R}$ is a smooth function defined in an open set of $\mathbb{R}^n$, then

(a) $\Delta f(x) = \frac{\partial^2 f}{\partial x_1^2} + \ldots + \frac{\partial^2 f}{\partial x_n^2} = \sum_{i=1}^{n} \langle \text{Hess}(f)_x(w_i), w_i \rangle$ where $\{w_1, \ldots, w_n\}$ is any orthonormal bases of $\mathbb{R}^n$ and Hess($f$) is the $n \times n$ hessian matrix of $f$.

(b) $\langle \text{Hess}(f) \nabla f, \nabla f \rangle = \frac{1}{2} \langle |\nabla f|^2, \nabla f \rangle$

(c) $\frac{d}{dt} \frac{\partial f(\alpha(t))}{\partial t} = \text{Hess}(f)_{\alpha(t)} \alpha'(t)$ for any smooth curve $\alpha : (a, b) \to V$.

Proof: (a) holds true because $\Delta f(x)$ is the trace of the matrix Hess($f$)$_x$, and the trace of a matrix is invariant under change of bases. (b) is a direct computation and (c) follows from the chain rule.

Before I proceed, I would like to thank Colciencias and Universidad del Valle for their financial support.

§2. Main result: In this section we will state and prove one of the main results of this paper.
**Theorem 2.1:** Let $M = \{ x \in V : f(x) = 0 \}$ where $f : V \to \mathbb{R}$ is a smooth function defined in an open set of $\mathbb{R}^n$ with $|\nabla f(x)| \neq 0$ for all $x \in M$. $M$ is minimal if and only if $|\nabla f|^2 \Delta f = \frac{1}{2} \langle \nabla |\nabla f|^2, \nabla f \rangle$ for every $x \in M$.

**Proof:** We are going to compute the mean curvature $H$ of $M$ in terms of the function $f$ and its partial derivatives. Let us start computing $\langle d\nu_x(v), v \rangle$ for any $v \in T_x M$. Let us take a smooth curve $\alpha : (-\epsilon, \epsilon) \to M$ such that $\alpha(0) = x$ and $\alpha'(0) = v$. We have that

$$
\langle d\nu_x(v), v \rangle = \langle \frac{d\nu(x(t))}{dt} \bigg|_{t=0}, v \rangle
= \langle \frac{d|\nabla f(x(t))|^{-1} \nabla f(x(t))}{dt} \bigg|_{t=0}, v \rangle
= \frac{d|\nabla f(x(t))|^{-1}}{dt} \bigg|_{t=0} \langle \nabla f(x), v \rangle + |\nabla f(x)|^{-1} \langle \frac{d|\nabla f(x(t))|}{dt} \bigg|_{t=0}, v \rangle
= 0 + |\nabla f(x)|^{-1} \langle \text{Hess}(f)_x v, v \rangle = |\nabla f(x)|^{-1} \langle \text{Hess}(f)_x v, v \rangle
$$

Now, if $\{v_1, \ldots, v_{n-1}\}$ is an orthonormal bases of $T_x M$, then by the equation (1) in section 1, we get that,

$$
H = - \sum_{i=1}^{n-1} \langle d\nu_x(v_i), v_i \rangle = -|\nabla f(x)|^{-1} \langle \text{Hess}(f)_x v_i, v_i \rangle
= |\nabla f(x)|^{-1} (-\Delta f(x) + \langle \text{Hess}(f)_x \nabla f(x), |\nabla f(x)|^{-1} \nabla f(x) \rangle)
= |\nabla f(x)|^{-1} (-\Delta f(x) + |\nabla f(x)|^{-2} \frac{1}{2} \langle \nabla |\nabla f|^2, \nabla f \rangle)
$$

Therefore we get that $M$ is minimal, if and only if, for every $x \in M$, we have that $|\nabla f(x)|^2 \Delta f(x) = \frac{1}{2} \langle \nabla |\nabla f|^2, \nabla f \rangle$. $\blacksquare$

**Example 2.1:** (Clifford minimal cones) Let $k$ and $l$ be two positive integers such that $k + l = n - 2$, and let $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be the function given by

$$
f(x) = f(x_1, \ldots, x_n) = k(x_1^2 + \ldots + x_{l+1}^2) - l(x_{l+2}^2 + \ldots + x_n^2)
$$

Let us check that $M_{lk} = f^{-1}(0)$ is a minimal hypersurface. A direct computation shows that

$$
\nabla f(x) = 2(kx_1, \ldots, kx_{l+1}, -lx_{l+2}, \ldots, -lx_n)
|\nabla f(x)|^2 = 4k^2(x_1^2 + \ldots + x_{l+1}^2) + 4l^2(x_{l+2}^2 + \ldots + x_n^2)
\nabla |\nabla f(x)|^2 = 8(k^2x_1, \ldots, k^2x_{l+1}, l^2x_{l+2}, \ldots, l^2x_n)
\frac{1}{2} \langle \nabla |\nabla f|^2, \nabla f \rangle = 8k^3(x_1^3 + \ldots + x_{l+1}^3) - 8l^3(x_{l+2}^3 + \ldots + lx_n^3)
\Delta f(x) = 2k(l + 1) - 2l(k + 1) = 2k(k - l)
$$
Therefore, we have that if \( x \in M \), i.e. if \( k(x_1^2 + \ldots + x_{l+1}^2) = l(x_{l+2}^2 + \ldots + x_n^2) \), then,

\[
|\nabla f(x)|^2 \Delta f = 2(k - l)(4k^2 - 4lk)(x_1^2 + \ldots + x_{l+1}^2) \\
= 8k(k^2 - l^2)(x_1^2 + \ldots + x_{l+1}^2) \\
= 8k^3x_1^2 + \ldots + 2x_{l+1}^2 - 8l^3(x_{l+2}^2 + \ldots + x_n^2) \\
= \frac{1}{2} \langle \nabla |\nabla f|^2, \nabla f \rangle
\]

We will say that \( M \) is a Clifford minimal cone if \( M = M_{kl} \) up to a rigid motion in \( \mathbb{R}^n \).

Let us assume now that \((N, g)\) is a riemannian \( n \) dimensional manifold, \( V \) is an open subset of \( N \) and \( f : V \to \mathbb{R} \) is a smooth function such that \( M = f^{-1}(0) \) is a hypersurface of \( N \), i.e. \( 0 \) is a regular value of \( f \). In this case we will denote by \( \text{Hess}(f)_x : T_x M \times T_x M \to \mathbb{R} \) the bilinear form defined by \( \text{Hess}(f)_x(v, w) = \langle D_v \nabla f, w \rangle \), where \( D \) is the Levi Civita connection on \( N \).

The exact same proof of the previous theorem gives us the following result.

**Theorem 2.2:** Let \( N \) be a riemannian manifold and let \( M = \{ x \in V : f(x) = 0 \} \) where \( f : V \to \mathbb{R} \) is a smooth function defined in an open set of \( N \) with \( |\nabla f(x)| \neq 0 \) for all \( x \in M \). \( M \) is minimal if and only if \( M \) is a Clifford minimal cone.

Before we prove this theorem we will need the following lemma

**Lemma 3.1:** let \( B \) be an invertible symmetric matrix with both, positive and negative eigenvalues. If \( C \) is a matrix that commutes with \( B \) such that \( \langle Cx, x \rangle = 0 \) always that \( \langle Bx, x \rangle = 0 \), then \( C = \lambda B \) for some real number \( \lambda \).

**Proof:** Since \( B \) and \( C \) commutes, after an orthogonal change of coordinates, we can assume that

\[
\langle Cx, x \rangle = 0 \quad \text{always that} \quad \langle Bx, x \rangle = 0,
\]

for every \( x \in \mathbb{R}^n \). Thus, we can write \( C = \lambda B \) for some real number \( \lambda \).
\[
C = \begin{pmatrix}
c_1 \\
\vdots \\
c_r \\
c_{r+1} \\
\vdots \\
c_n
\end{pmatrix}
\quad \quad \quad
B = \begin{pmatrix}
b_1 \\
\vdots \\
b_r \\
b_{r+1} \\
\vdots \\
b_n
\end{pmatrix}
\]

where 0 < r < n, \(b_1, \ldots, b_r\) are positive real numbers and \(b_{r+1}, \ldots, b_n\) are negative real numbers. Let us denote \(e_1 = (1,0,\ldots,0), \ldots, e_n = (0,\ldots,0,1)\) the canonical bases for \(\mathbb{R}^n\) and for \(1 \leq i \leq r\) and \(r < j \leq n\) we will denote \(x_{ij} = \sqrt{-b_j}e_i + \sqrt{b_i}e_j\). Notice that \(\langle Bx_{ij}, x_{ij} \rangle = 0\), therefore \(\langle Cx_{ij}, x_{ij} \rangle = 0\), i.e.

\[-b_jc_i + b_ic_j = 0\]

from the equation above we get that \(c_i = \frac{c_n}{b_n}b_i\) for \(1 \leq i \leq r\) and for \(r < j \leq n\) we have, using the expression for \(c_1\), that

\[c_j = \frac{c_1}{b_1}b_j = \frac{c_n}{b_n}b_1\frac{1}{b_1}b_j = \frac{c_n}{b_n}b_j\]

Therefore \(C = \frac{c_n}{b_n}B\), this completes the proof. \(\blacksquare\)

Now we are ready to prove Theorem 3.1.

**Proof of Theorem 3.1:** A Direct computation shows that \(\nabla f(x) = 2Bx\). Since we are assuming that 0 is a regular value, then \(\nabla f(x) \neq 0\) for all \(x \in M\), in particular \(Bx_0 \neq 0\) for every \(x_0 \neq 0\), because if \(Bx_0 = 0\) for some \(x_0 \neq 0\) we would have that \(x_0 \in M\) and \(\nabla f(x_0) = 0\). Therefore \(B\) is an invertible matrix. We also have that, since \(M \neq \emptyset\), then \(B\) must have positive eigenvalues and negative eigenvalues. We have that

\[|\nabla f(x)|^2 = \langle 2Bx, 2Bx \rangle = 4\langle Bx, Bx \rangle = 4\langle B^2x, x \rangle\]

Since \(B^2\) is symmetric, then \(\nabla |\nabla f|^2(x) = 8B^2x\). A direct computation shows that \(\Delta f = 2\text{trace}(B)\). Using Theorem 2.1, we have that \(M\) is minimal if and only if \(|\nabla f|^2\Delta f = \frac{1}{2} \langle \nabla |\nabla f|^2, \nabla f \rangle\) for every \(x \in M\), i.e. if, for every \(x \neq 0\) such that \(\langle Bx, x \rangle = 0\) we have that

\[4\langle B^2x, x \rangle(2\text{trace}(B)) = \frac{1}{2} \langle 8B^2x, 2Bx \rangle = 8\langle B^3x, x \rangle\]

In others words, if we define \(C = \text{trace}(B)B^2 - B^3\), we have that \(M\) is minimal if and only if \(\langle Cx, x \rangle = 0\) for every \(x\) such that \(\langle Bx, x \rangle = 0\). Using Lemma 3.1 we concluded that there exists a real number \(a\) such that \(\text{trace}(B)B^2 - B^3 = aB\). Since \(B\) is an invertible matrix, we get that \(B\) satisfies the following polynomial equation
\[ B^2 - \text{trace}(B)B + aI = 0 \]  

(2)

Therefore \( B \) can only have two eigenvalues. Since \( B \) has negative and positive eigenvalues, we can assume that the eigenvalues of \( B \) are \( \lambda_1 > 0 \) with multiplicity \( r \geq 1 \) and \( \lambda_2 < 0 \) with multiplicity \( n - r \geq 1 \). Notice that \( \text{trace}(B) = r\lambda_1 + (n - r)\lambda_2 \). The equation (2) is equivalent to the following system of equations for \( \lambda_1, \lambda_2 \) and \( a \).

\[
\begin{align*}
\lambda_1^2 - (r\lambda_1 + (n-r)\lambda_2)\lambda_1 + a &= (1-r)\lambda_1^2 - (n-r)\lambda_1\lambda_2 + a = 0 \\
\lambda_2^2 - (r\lambda_1 + (n-r)\lambda_2)\lambda_2 + a &= -(n-r-1)\lambda_2^2 - r\lambda_1\lambda_2 + a = 0 
\end{align*}
\]

combining these two equations we get

\[
(1-r)\lambda_1^2 - (n-2r)\lambda_1\lambda_2 + (n-r-1)\lambda_2^2
\]

(3)

From this equation we get that \( r = 1 \) or \( r = n - 1 \) implies that \( \lambda_1 = \lambda_2 \) which is impossible because \( \lambda_1\lambda_2 < 0 \). Therefore \( 1 < r < n - 1 \). From equation (3) we get that \( t = \frac{\lambda_2}{\lambda_1} \) satisfies the equation

\[
(1-r) - (n-2r)t + (n-r-1)t^2
\]

Therefore \( t = 1 \) or \( t = \frac{r-1}{n-r} \), since we have that \( t \) must be negative, then \( t \) cannot be 1. Therefore up to a constant we may take \( \lambda_1 = n-r-1 \) and \( \lambda_2 = r-1 \). i.e. Up to a rigid motion \( f(x) = \langle Bx, x \rangle \) must be a multiple of the function given in the example 2.1. This implies that \( M \) must be a Clifford minimal cone. 

**Remark on the construction of minimal hypersurfaces using homogeneous polynomials of degree \( k \):** Let \( f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R} \) be a homogeneous polynomial of degree \( k \) such that \( f^{-1}(0) = M \) is not empty and such that for every \( x \in M, \nabla f(x) \neq 0 \). By theorem 2.1 we have that \( M \) is minimal if and only if

\[
g(x) = |\nabla f|^2 \Delta f - \frac{1}{2} (\nabla|\nabla f|^2, \nabla f) = 0
\]

(4)

for every \( x \) such that \( f(x) = 0 \). Notice that the left hand side of the equation (4) is a homogeneous polynomial of degree \( 3k - 4 \), also notice that if \( g(x) = h(x)f(x) \) for some homogeneous polynomial \( h \) of degree \( 2k - 4 \), then \( M \) will be minimal.

It is easy to prove that the veracity of the conjecture:

**Conjecture:** Let \( f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R} \) be a homogeneous polynomial of degree \( k \) such that \( f^{-1}(0) = M \) is not empty and such that for every \( x \in M, \nabla f(x) \neq 0 \). If \( g(x) \) is a polynomial of degree \( m \) with \( m \geq k \) such that \( g(x) = 0 \) for every \( x \in M \), then there exists a homogeneous polynomial \( h \) of degree \( m - k \) such that \( g(x) = h(x)f(x) \).
implies the following result:

“Let $M = f^{-1}(0) \neq \emptyset$ where $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be a homogeneous polynomial. $M$ is minimal if and only if

$$|\nabla f|^2 \Delta f - \frac{1}{2} (\nabla |\nabla f|^2, \nabla f) = hf$$

for some homogeneous polynomial $h$”.

So far the only known examples of these minimal hypersurfaces are the isoparametric minimal hypersurfaces, the degree of $f$ in this examples are $k = 1, 2, 3, 4, 6$. Notice that Lemma 2.1 proves the conjecture when $k = 2$.

**Bibliography**