Distribution of minimal varieties in spheres in terms of the coordinate functions

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ABSTRACT: Let $M$ be a compact $k$-dimensional riemmanian manifold minimally immersed in the unit $n$-dimensional sphere $S^n$. It is easy to show that for any $p \in S^n$ the boundary of the geodesic ball in $S^n$ with radius $\frac{\pi}{2}$ and center at $p$ (in this case this boundary is an equator) must intercept the manifold $M$. When the codimension is 1, i.e. $k = n - 1$, it is known that the ricci curvature, is not greater than 1. We will prove that if the ricci curvature is not greater than $1 - \frac{\alpha^2}{n-2}$, then the boundary of every geodesic ball with radius $\cot^{-1}(\alpha)$ must intercept the manifold $M$. We give examples of manifolds for which the radius $\cot^{-1}(\alpha)$ is optimal. Next, for any codimension, i.e. for any $M^k \subset S^n$, we find a number $r_1$ that depends only on $n$ such that for any collection of $n + 1$ points $\{p_i\}_{i=1}^{n+1}$ in $S^n$ that constitutes an orthonormal basis of $R^{n+1}$, the union of the boundaries of the geodesic balls with radius $r_1$ and center $p_i$, $i = 1, 2, \ldots n + 1$, must intercept the manifold $M$.

§1 Introduction and preliminaries.

Let $M$ be a compact, oriented minimal hypersurface immersed in the $n$-dimensional unit sphere $S^n$. Let $\nu$ be a unit normal vector field along $M$. for any tangent vector $v \in T_mM$, $m \in M$, the shape operator $A$ is given by $A(v) = -\bar{D}_v \nu$, where $\bar{D}$ denotes the Levi Civita connection in $R^{n+1}$. With the same notation, for any tangent vector field $W$, the Levi Civita connection on $M$ is given by $D_vW = (\bar{D}_vW)^T$ where $(\ )^T$ denotes the orthogonal projection from $R^{n+1}$ to $T_mM$. For a function $f : M \rightarrow R$, $\nabla f$ will denote the gradient of $f$. For any pair of vectors $v, w \in T_mM$ the hessian of $f$ is given by $H(f)(v, w) = \langle D_v \nabla f, w \rangle$, where $\langle \ , \ \rangle$ denotes the inner product in $R^{n+1}$. The Laplacian of $f$ is given by $\Delta(f) = \sum_{i=1}^{n+1} H(f)(v_i, v_i)$, where $\{v_i\}_{i=1}^{n+1}$ is an orthonormal basis of $T_mM$.

For a given $w \in R^{n+1}$ let us define the functions $l_w : M \rightarrow R$ and $f_w : M \rightarrow R$ by $l_w(m) = \langle m, w \rangle$ and $f_w(m) = \langle \nu(m), w \rangle$. Clearly the function $l_w$ is the restriction to $M$ of the linear function in $R^{n+1}$ whose gradient is the constant vector $w$, therefore the gradient of the function $l_w$ at $m \in M$ is the projection of the vector $w$ to $T_mM$, i.e.
\[ \nabla l_w = w^T = w - l_w(m)m - f_w(m)\nu(m) \]

Let \( v_i, v_j \) be two vectors in \( T_m M \). The hessian of \( l_w \) at \( m \in M \), is given by:

\[
H(l_w)(v_i, v_j) = \langle D_{v_i} \nabla l_w, v_j \rangle = \langle \bar{D}_{v_i} \nabla l_w, v_j \rangle = -l_w(v_i, v_j) - f_w(\bar{D}_{v_i} \nu, v_j) = -l_w(v_i, v_j) + f_w(A(v_i), v_j) \quad (1.1)
\]

From the equation above and the fact that \( A \) is traceless (minimality of \( M \)) we get that:

\[
\Delta l_w = -(n - 1)l_w \quad (1.2)
\]

**Remark 1.3** As a corollary of Equation (1.2) we get that every coordinate function \( l_w \) must change sign, therefore the boundary of every geodesic ball with radius \( \frac{\pi}{2} \) must intersect \( M \).

Given any non-equatorial compact minimal hypersurface in \( S^n \) we know that there exists a radius \( r, \quad r < \frac{\pi}{2} \), such that \( M \) must intersect the boundary of every geodesic ball in \( S^n \) with radius \( r \). Let \( \gamma_M \) be the minimum \( r \) with the property above. In section 2 we will use the expression for the hessian of the coordinate function \( l_w \) to find an upper bound for \( \gamma_M \), namely we will show:

**Theorem 1.4:** Let \( M^{n-1} \) be a minimal hypersurface immersed in \( S^n \) and let \( \{\lambda_i(m)\}_{i=1}^{n-1} \) be the eigenvalues of the shape operator at \( m \in M \). Define \( \bar{\alpha}(m) = \min\{|\lambda_i(m)|, \quad i = 1 \ldots n - 1\} \) and let \( \alpha \) be the minimum over \( M \) of the function \( \bar{\alpha} \). If \( r_0 \) satisfies that \( \cot(r_0) = \alpha \) and \( 0 < r_0 \leq \frac{\pi}{2} \), i.e. \( r_0 = \cot^{-1}(\alpha) \), then the boundary of every geodesic ball in \( S^n \) with radius \( r_0 \) must intersect \( M \).

Notice that if \( \alpha = 0 \), then Theorem 1.4 reduces to Remark 1.3. A direct computation shows that if \( M \) is the minimal Clifford hypersurface, \( M = \{(x, y) \in \mathbb{R}^{s+1} \times \mathbb{R}^{s+1} : \|x\|^2 = \|y\|^2 = \frac{1}{2}\} \), then for any \( r < \frac{\pi}{2} \), the boundary of the geodesic ball with center at \((1, 0, \ldots, 0)\) does not intersect \( M \), therefore \( \gamma_M \geq \frac{\pi}{4} \). The principal eigenvalues of the shape operator \( A \) of \( M \) are either 1 or \(-1\) everywhere, then \( \alpha = 1 \) in this case and we get that \( \frac{\pi}{4} = \cot^{-1}(1) \geq \gamma_M \geq \frac{\pi}{4} \).

This example shows that the estimate in Theorem 1.4 is sharp.
Let us rewrite Theorem 1.4 in terms of the curvature of $M$. Denote by $R$ and Ricc the curvature tensor and the Ricci curvature of $M$ respectively. The Gauss equation states that

$$
\langle R(v,w)v, w \rangle = \langle v, v \rangle \langle w, w \rangle - \langle v, w \rangle \langle v, w \rangle + \langle A(v), v \rangle \langle A(w), w \rangle - \langle A(w), v \rangle \langle A(v), w \rangle
$$

Therefore, if $\{v_i\}_{i=1}^{n-1}$ is an orthonormal basis of $T_m M$ we have

$$
\text{Ricc}(v) = \frac{1}{n-2} \left( \sum_{i=1}^{n-1} \langle R(v,v_i)v, v_i \rangle \right)
= \frac{\sum_{i=1}^{n-1} ((\langle v, v_i \rangle \langle e_i, e_i \rangle - \langle e_i, v \rangle \langle e_i, v \rangle + \langle A(v), v \rangle \langle A(e_i), e_i \rangle - \langle A(e_i), v \rangle \langle A(v), e_i \rangle) }{n-2}
= \frac{(n-1)|v|^2 - |v|^2 - |A(v)|^2}{n-2}
= |v|^2 - \frac{|A(v)|^2}{n-2}
$$

By the equation above, we get that another way to define $\alpha$ is given by,

$$
\max_{v \in T^1 M} \text{Ricc}(v) = 1 - \frac{\alpha^2}{n-2}
$$

where $T^1 M = \{ v \in T_m M : m \in M$ and $|v| = 1 \}$. By the observations made above we get:

**Corollary 1.5:** Let $M \subset S^n$ be a minimal hypersurface. If $\text{Ricc}(v) \leq 1 - \frac{\alpha^2}{n-2}$ for every $v \in T^1 M$, then the boundary of every geodesic ball in the sphere with radius $\cot^{-1}(\alpha)$ must intersect $M$.

The result in the previous corollary needs the minimality condition. To see this, it is enough to look at the following family of flat surfaces in $S^3$ given by

$$
M_{r_1, r_2} = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 : |x|^2 = r_1^2, |y|^2 = r_2^2, r_1^2 + r_2^2 = 1$ and $r_1 \leq r_2 \}
$$

A direct computation shows that $\gamma_{M_{r_1, r_2}} = \sin^{-1}(r_2)$, we also have that $\alpha$ is 1 because all these surfaces are flat, therefore if $r_1 < r_2$ then $\gamma_{M_{r_1, r_2}} > \frac{\pi}{4}$; hence $\frac{\pi}{4} = \cot^{-1}(\alpha)$ is not an upper bound for $\gamma_{M_{r_1, r_2}}$. The examples above show us that among all euclidean product
of circles in $S^3$, the minimal Clifford tori are the ones that best makes the work of “trying to be as close of every point in $S^3$ as possible, in other words, the minimal Clifford torus minimize $\gamma_{M_{r_1 r_2}}$ in the family $M_{r_1 r_2}$.

Our second result states that if $M$ is a minimal variety in $S^n$, then at least one of its coordinates functions must take the value $-(1 - \sqrt{\frac{n-2}{n+1}})$ at some point. Namely we will show:

**Theorem 1.6** Let $M^k$ be a minimal $k$-dimensional manifold immersed in the $n$-dimensional unit sphere $S^n$. Then for every orthonormal basis of $\mathbb{R}^{n+1}$, $\{p_i\}_{i=1}^{n-1}$, for some $i$, $M$ must intersect the boundary of geodesic ball with center at $p_i$ and radius $\cos^{-1}(1 - \sqrt{\frac{n-2}{n+1}})$.

Before I proceed, I would like to thank Professor Bruce Solomon for his lessons on mathematics and his comments on this paper. I would like also to thank Professor Peter Li for meeting with me to discuss mathematics, one of his comments motivated the idea for Theorem 1.4.

§2 Proof of the theorems.

We start this section stating and proving Theorem 1.4. This result is a consequence of Equation (1.1) for the hessian of the coordinate functions. Notice that in both of the theorems, 1.4 and 1.5, we may assume that our manifold $M$ is orientable, since otherwise the results follow by applying the theorems to the double covering of $M$.

**Theorem 1.4:** Let $M^{n-1}$ be a minimal hypersurface immersed in $S^n$ and let $\{\lambda_i(m)\}_{i=1}^{n-1}$ be the eigenvalues of the shape operator at $m \in M$. Define $\bar{\alpha}(m) = \min\{|\lambda_i(m)|, \ i = 1 \ldots n - 1\}$ and let $\alpha$ be the minimum over $M$ of the function $\bar{\alpha}$. If $r_0$ satisfies that $\cot(r_0) = \alpha$ and $0 < r_0 \leq \frac{\pi}{2}$, i.e. $r_0 = \cot^{-1}(\alpha)$, then the boundary of every geodesic ball in $S^n$ with radius $r_0$ must intersect $M$.

**Proof:** Since the result is trivial when $M$ is totally geodesic we will assume that this is not the case. Notice that it is enough to prove that for every $v \in S^n$ the minimum of the coordinate function $l_v : M \rightarrow \mathbb{R}$ over $M$ is less than or equal to $\frac{\bar{\alpha}}{\sqrt{1+\alpha^2}}$. Let $m_0$ be a point in $M$ where the function $l_v$ reaches its minimum. Since $M$ is not an equator we have that $l_v(m_0) < 0$. We need to show that $l_v(m_0) \leq \frac{\bar{\alpha}}{\sqrt{1+\alpha^2}}$ or equivalently $|l_v(m_0)| \geq \frac{\alpha}{\sqrt{1+\alpha^2}}$. Since $m_0$ is a critical point of the function $l_v$ we have $\nabla l_v = 0$, therefore,
Let \( \{v_i\}_{i=1}^{n-1} \) be an orthonormal basis of \( T_{m_0}M \) that diagonalizes the shape operator \( A \) at \( m_0 \). Since \( m_0 \) is a minimum of \( l_v \) we get for \( i = 1 \ldots n-1 \) that,

\[
0 \leq H(l_v)(v_i, v_i) = -l_v(m_0)\langle v_i, v_i \rangle + f_v(m_0)\langle A(v_i), v_i \rangle = -l_v(m_0) + f_v(m_0)\lambda_i \tag{2.3}
\]

Since \( \sum_{i=1}^{n-1} \lambda_i = 0 \) we can pick \( k \) such that \(-f_v(m_0)\lambda_k\) is not negative. Using the definition of \( \alpha \), equation (2.1) and the inequalities (2.3) we get:

\[
|l_v(m_0)| = -l_v(m_0) \geq -f_v(m_0)\lambda_k = | -f_v(m_0)| |\lambda_k| \geq \alpha \sqrt{1-l_v(m_0)^2}
\]

Finally, from the inequality above we can easily deduce the inequality we were looking for:

\[
|l_v(m_0)| \geq \frac{\alpha}{\sqrt{1+\alpha^2}} = \beta. \quad \blacksquare
\]

A direct computation shows that the formula for the laplacian of the coordinate functions, (1.2), hold true for any codimension, i.e. we have that if \( M^k \) is a \( k \)-dimensional manifold minimally immersed in \( S^n \) then \(-\Delta l_w = k l_w\).

Now we will prove our second result. For the reader’s convenience we will restate this theorem:

**Theorem 1.6** Let \( M^k \) be a minimal \( k \)-dimensional manifold immersed in the \( n \)-dimensional unit sphere \( S^n \). Then for every orthonormal basis of \( \mathbb{R}^{n+1}, \{p_i\}_{i=1}^{n-1}, \) for some \( i, M \) must intersect the boundary of geodesic ball with center at \( p_i \) and radius \( \cos^{-1}(1 - \sqrt{\frac{n-2}{n+1}}) \).

**Proof:** Notice that it is enough to prove that a least one of the coordinates functions \( l_{p_i} \) takes a value less than or equal to \(-1 - \sqrt{\frac{n-2}{n+1}}\). We will proceed by contradiction. Let us assume that all the functions \( l_i = l_{p_i} \) are greater than or equal to \(-1 - \sqrt{\frac{n-2}{n+1}}\). Therefore the vector fields \( X_i = (1 + l_i)^{-1}\nabla l_i \) are well defined. Let us compute the divergence of \( X_i \):

\[
\text{div}(X_i) = -(1 + l_i)^{-2}\|\nabla l_i\|^2 + (1 + l_i)^{-1}(-k l_i) = \frac{k}{2}(-1 + (1 + l_i)^{-2}(1 - \frac{2}{k}\|\nabla l_i\|^2 - l_i^2))
\]
Since \((1 + l_i)^{-2} \leq (1 - (1 - \sqrt{\frac{n+2}{n+1}}))^{-2} = \frac{n+1}{n+2}\) by assumption and \(1 - \frac{2}{k} \|\nabla l_i\|^2 - l_i^2 > 0\), we get from the expression above after using the divergence theorem that,

\[
0 = \int_M (-1 + (1 + l_i)^{-2}(1 - \frac{2}{k} \|\nabla l_i\|^2 - l_i^2)) < \int_M (-1 + \frac{n+1}{n-2}(1 - \frac{2}{k} \|\nabla l_i\|^2 - l_i^2)).
\]

Notice that \(\sum_{i=1}^{n+1} l_i^2 = 1\) and by Stoke’s theorem \(\int_M \|\nabla l_i\|^2 = k \int_M l_i^2\). Then, if we sum the inequalities above from \(i = 1\) to \(i = n + 1\) we get,

\[
0 < \sum_{i=1}^{n+1} \int_M (-1 + \frac{n+1}{n-2}(1 - \frac{2}{k} \|\nabla l_i\|^2 - l_i^2)) = \int_M (- (n + 1) + \frac{n+1}{n-2}(n + 1 - 2 - 1)) = 0
\]

This contradiction proves the theorem. □

**Remark:** The case \(n = 3\) in the theorem above is also a consequence of the fact that any two minimal surfaces in \(S^3\) must intersect [F]. For any minimal surface \(M\) in \(S^3\) the surface \(-M = \{p \in S^3 : -p \in M\}\) is also minimal. If \(m_0 \in -M \cap M\) then one can check that if \(\Gamma\) is defined as above, either \(m_0\) or \(-m_0\) intersects \(\Gamma\).

**REFERENCES**